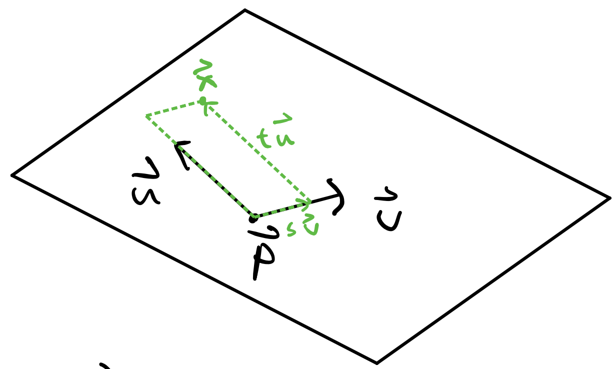
Plane (§12.5)

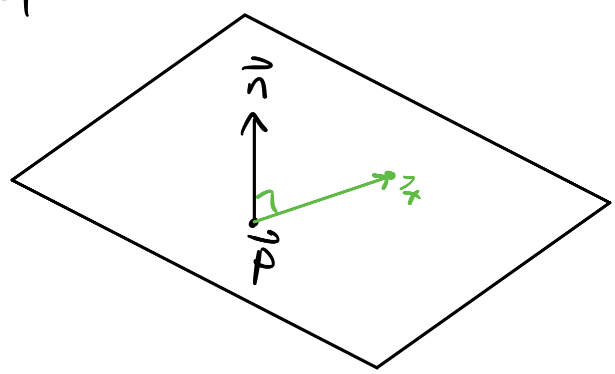
A plane in $\mathbb{R}^3$ can be described by:



(i). a point $\vec{p}$ and two linearly independent (direction) vectors $\vec{u}$, $\vec{v}$:

$$\{\vec{x} = \vec{p} + t\vec{u} + s\vec{v} \mid t, s \in \mathbb{R}\}$$

(ii) a point $\vec{p}$ and a normal vector $\vec{n}$ to the plane:



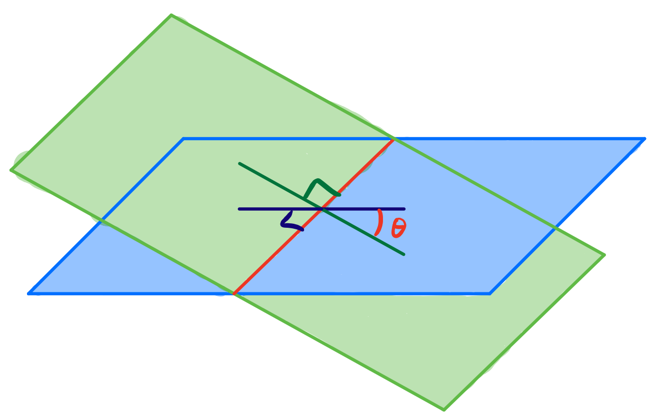
$$\{\vec{x} \in \mathbb{R}^3 \mid 0 = \vec{n} \cdot (\vec{x} - \vec{p})\}$$

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

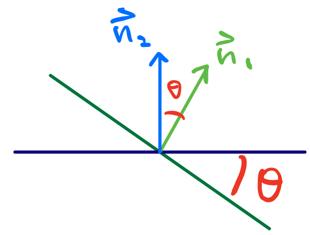
$$\Longleftrightarrow \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

$$ax + by + cz = d, \quad \vec{n} = (a, b, c)$$

Two non-parallel planes will intersect at a line, and angle between the two planes is the acute angle between two lines, lying on the planes, perpendicular to the line of intersection.



The angle can be calculated using normal vectors of the planes:



$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|}\right)$$

e.g. Two planes : $3x - 6y - 2z = 15$

$2x + y - 2z = 5$

has normal vectors : $\vec{n}_1 = (3, -6, -2)$

$\vec{n}_2 = (2, 1, -2)$

correspondingly, and the angle between two planes is :

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|}\right)$$

$$= \cos^{-1}\left(\frac{4}{7 \cdot 3}\right)$$

$$\approx 1.38$$

The steps can be generalised to any dimensional Euclidean spaces.

§12.5

In Exercises 53–56, find the point in which the line meets the plane.

53. $\underbrace{x = 1 - t, \quad y = 3t, \quad z = 1 + t;}_{\text{a line}} \quad \underbrace{2x - y + 3z = 6}_{\text{plane.}}$

Intersect if

$$2(1-t) - (3t) + 3(1+t) = 6$$

$$\Leftrightarrow \quad 2 - 2t + 3 = 6$$

$$\Leftrightarrow \quad t = -1/2$$

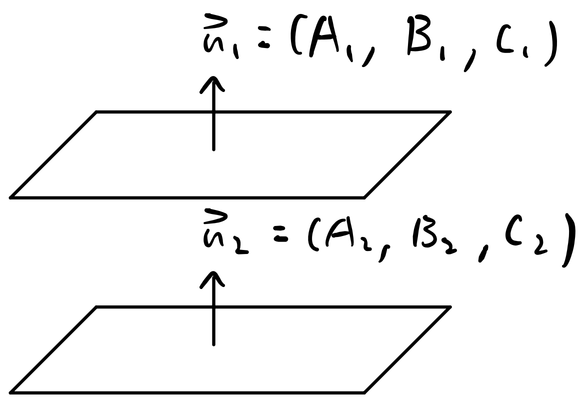
$$\Leftrightarrow \quad x = 3/2, \quad y = -3/2, \quad z = 1/2.$$

68. How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? Perpendicular? Give reasons for your answer.

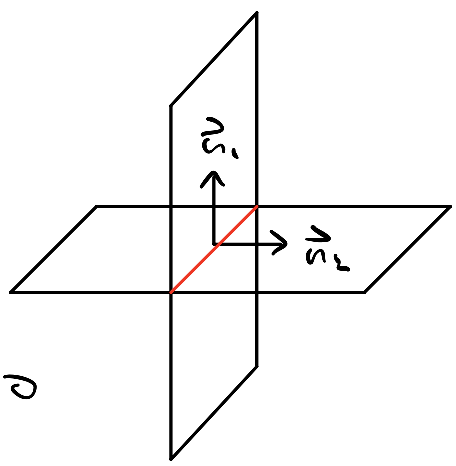
Two planes are parallel if their normal vectors are pointing the same or opposite direction

i.e. $\vec{n}_1 = a\,\vec{n}_2, \quad a \in \mathbb{R}$

or $\vec{n}_1 \times \vec{n}_2 = \vec{0}$.



They are perpendicular if their normal vectors are perpendicular. i.e. $\vec{n}_1 \cdot \vec{n}_2 = 0$

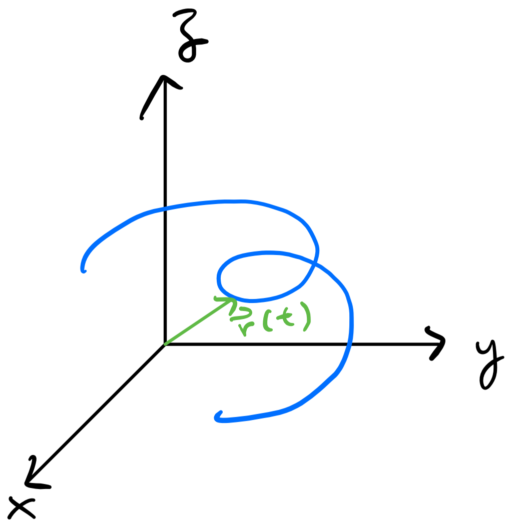


<u>Curves</u> (§13.1)

a curve can be understood as
a particle's path:

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

$t \in I \subseteq \mathbb{R}$
(Interval)



<u>Def</u>

$\vec{r}$ is said to differentiable at t

if $\displaystyle\lim_{h\to 0} \frac{\vec{r}(t+h) + \vec{r}(t)}{h}$ exists.

<u>Remark</u>

If f, g, h are differentiable
at t, then $\vec{r}$ is differentiable at t
and $\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$.

Differentiation involving Dot / Cross product.

$\vec{r}_1(t)$, $\vec{r}_2(t)$ differentiable ,

$$\frac{d}{dt}\left(\vec{r}_1(t)\cdot\vec{r}_2(t)\right)$$

$$= \frac{d}{dt}\left(f_1(t)f_2(t) + g_1(t)g_2(t) + h_1(t)h_2(t)\right)$$

$$\begin{aligned} = \ & f_1'(t)f_2(t) + f_1(t)f_2'(t) \\ + \ & g_1'(t)g_2(t) + g_1(t)g_2'(t) \\ + \ & h_1'(t)h_2(t) + h_1(t)h_2'(t) \end{aligned}$$

$$= \vec{r}\,'_1(t)\vec{r}_2(t) + \vec{r}_1(t)\vec{r}_2\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right)$$

$$= \lim_{h\to 0}\frac{\vec{r}_1(t+h)\times\vec{r}_2(t+h) - \vec{r}_1(t)\times\vec{r}_2(t)}{h}$$

$$= \lim_{h\to 0}\frac{\vec{r}_1(t+h)\times\vec{r}_2(t+h) - \vec{r}_1(t+h)\times\vec{r}_2(t)}{h}$$

$$+ \lim_{h\to 0}\frac{\vec{r}(t+h)\times\vec{r}_2(t) - \vec{r}_1(t)\times\vec{r}_2(t)}{h}$$

$$= \lim_{h\to 0}\vec{r}_1(t+h)\times\lim_{h\to 0}\frac{\vec{r}_2(t+h)-\vec{r}_2(t)}{h}$$

$$+ \lim_{h\to 0}\frac{\vec{r}_1(t+h)-\vec{r}_1(t)}{h}\times\lim_{h\to 0}\vec{r}_2(t)$$

$$= \vec{r}_1(t)\times\vec{r}_2\,'(t) + \vec{r}_1\,'(t)\times\vec{r}_2(t)$$

28. Derivatives of triple scalar products

a. Show that if **u**, **v**, and **w** are differentiable vector functions of t, then

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}.$$

b. Show that

$$\frac{d}{dt}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right).$$

(a).
$$\frac{d}{dt}[\vec{u}\cdot(\vec{v}\times\vec{w})]$$
$$= \vec{u}'\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot(\vec{v}\times\vec{w})'$$
$$= \vec{u}'\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot(\vec{v}'\times\vec{w}) + \vec{u}\cdot(\vec{v}\times\vec{w}')$$

(b). By (a),
$$\frac{d}{dt}\left[\vec{r}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)\right]$$
$$= \underbrace{\frac{d\vec{r}}{dt}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)}_{(*)} + \underbrace{\vec{r}\cdot\left(\frac{d^2\vec{r}}{dt^2}\times\frac{d^2\vec{r}}{dt^2}\right)}_{(**)} + \vec{r}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3}\right)$$

(*) : since $\vec{u}\perp\vec{u}\times\vec{v}$ $\forall\,\vec{u},\vec{v}$,

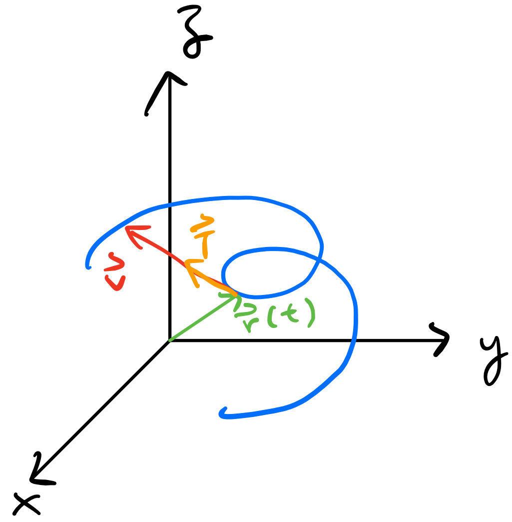
hence $=0$

(**) : $\vec{v}\times\vec{v}=\vec{0} \Rightarrow \vec{u}\cdot(\vec{v}\times\vec{v})=0$ $\forall\,\vec{u},\vec{v}$,

hence $=0$.

Curvature (§ 13.4)

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$



velocity:

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$$

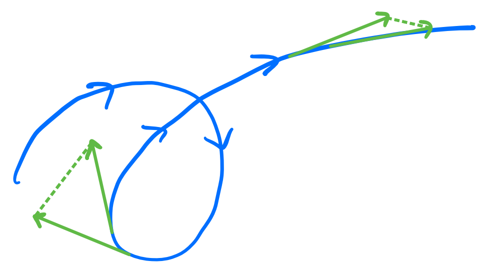
speed: $\|\vec{v}(t)\|$

direction:

$$\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$$

The curvature of the curve is:

$$\kappa = \frac{1}{\|\vec{v}\|}\left\|\frac{d\vec{T}}{dt}\right\|$$

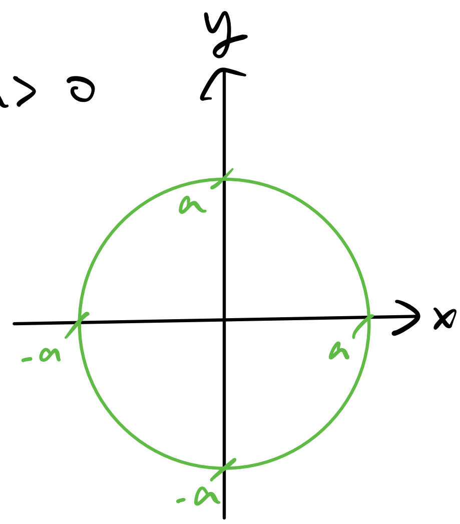


rapid change in direction

$\Rightarrow$ κ large

e.g.

$$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}, \quad a > 0$$



$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= -a\sin t\,\vec{i} + a\cos t\,\vec{j}$$

$$\|\vec{v}\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2}$$

$$= a$$

$$\vec{T} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\left\| \frac{d\vec{T}}{dt} \right\| = \sqrt{(-\cos t)^2 + (-\sin t)^2}$$

$$= 1$$

$$\therefore \quad \kappa = \frac{1}{\|\vec{v}\|} \left\| \frac{d\vec{T}}{dt} \right\|$$

$$= \frac{1}{a}.$$